PHYSICAL REVIEW D VOLUME 12, NUMBER 4 15 AUGUST 1975

Chiral symmetry and pion condensation. II. General formalism

David K. Campbell*
Theoretical Division, Los Alamos Scientific Laboratory, Los Alamos, New Mexico 87544
and The Institute for Advanced Study, Princeton, New Jersey 08540

Roger F. Dashen and Jamal T. Manassah†
The Institute for Advanced Study, Princeton, New Jersey 08540
(Received 11 December 1974)

The role of chiral symmetry in pion condensation was explored through model calculations in the preceding paper. Here we present model-independent results which justify and sometimes go beyond the previous work. Our methods are based on functional techniques used in field theory. The paper is not self-contained and is meant to be read in conjunction with the preceding one. In the course of this work we were led to look at low-energy π-N interactions. In a brief survey, it is pointed out that they are more complicated than is commonly assumed.

I. INTRODUCTION

In paper I,[1] we discussed pion condensation[2] using a chiral-symmetry approach inspired by the σ model. The σ model cannot be taken seriously as a realistic model of π-N and N-N interactions. If one were able to compute the binding energy of nuclear matter or π-N phase shifts in this model it would no doubt fail miserably.

Fortunately, many of the *qualitative* results of paper I do not depend on the details of the model. They are consequences of the fact that the σ model contains pions and nucleons, and that it has an approximate chiral SU(2)⊗SU(2) symmetry.[3]

The strong interactions are believed to respect an abstract SU(2)⊗SU(2) symmetry just as they respect isospin or SU(3). Those features of pion condensation which can be understood from chiral-symmetry considerations alone should therefore be present in any realistic calculation. Since they are consequences only of symmetry they can be abstracted from any model which respects the symmetry.

On the other hand, there are many features of pion condensation in general and of our model calculations in particular that are not just consequences of chiral symmetry. As was pointed out in the preceding paper, the inclusion of the $N^*(1236)$ as well as nuclear forces is necessary in order to get realistic results. Such strong-interaction phenomena are compatible with and indeed respect chiral symmetry. However, only a few of their properties are predicted by the symmetry. A realistic calculation requires considerable further input. In particular, there are questions about the low-energy π-N interaction which is only partially determined by chiral symmetry.

In any quantitative theory of pion condensation one is faced with a difficult many-body problem. Chiral-symmetry ideas are helpful in that they allow one to set up this problem in a systematic way, but they do not help much in solving it.

These caveats aside, we feel that the idea of chiral symmetry is basic to a fundamental understanding of pion condensation. In a previous letter[4] we have shown how to apply the ideas of chiral symmetry to pion condensation in the absence of fermions in a model-independent way. In this paper we carry out the same program for the physically interesting case where fermions are present. In doing so, we will justify many of the calculations of paper I. We will also see the limits of what can be learned from chiral-symmetry considerations alone.

In Sec. II we define some properties of the generating functionals of field theory and we argue that the energy functional $E_{\rm eff}(\varphi^{\rm cl})$ can be conveniently written as

$$E_{\rm eff} = E^v + E^H + E^c , \tag{1.1}$$

where v, H, c refer, respectively, to appropriately defined vacuum, Hartree, and correlation energies.

In Sec. III we compute the vacuum and Hartree energies in a model-independent way. In the k, μ, θ notation of the preceding paper, we show that

$$\frac{E^v}{V} = (f_\pi{}^2)[\tfrac{1}{2}(\vec{k}^2 - \mu^2)\sin^2\theta + m_\pi{}^2(1-\cos\theta)] + O(\mathfrak{L}_{\rm SB}{}^2, \vec{k}^4, \mu^4), \tag{1.2}$$

where the "standard" $(\frac{1}{2}, \frac{1}{2})$ symmetry breaking has been assumed, and that

$$E^H = E_0^H + N_B\left\{\Delta M(\theta^2) - f_\pi{}^2\frac{\sin^2\theta}{2}\left[(d_1+d_2)\mu^2 - d_2\vec{k}^2\right]\right\} + O(\mathcal{L}_{SB}{}^2, \mu^4, \vec{k}^4), \tag{1.3}$$

where E_0^H refers to the Hartree energy of the σ model, ΔM is related to the σ term in π-N scattering, and d_1 and d_2 are the Taylor expansion coefficients for an on-shell π-N scattering amplitude.

The correlation energy E^c is a many-body term about which chiral symmetry has little to say. It depends on nuclear forces, nucleon-nucleon correlations, and so on. The meaning of chiral symmetry with respect to nuclear forces was discussed in Sec. V of the preceding paper. Except for the possibility of estimating E^c by computing at $\theta = \pi/2$, where some simplification occurs, our formalism yields little new information about E^c.

In Sec. IV we examine in a general way the spontaneous breakdown of certain continuous symmetries in the presence of pion condensation.

In Sec. V we review some salient points about the experimental π-N interactions. It is pointed out there that this interaction is more complicated than is commonly thought and that simple models can be quite misleading.

In Appendix A we review the evaluation of Feynman integrals for $\mu \neq 0$ and $T = 0$.

In Appendix B we examine the spontaneous breakdown of discrete symmetries (P, CP, and T).

We would like at this point to stress that this paper is meant to be read along with or following the preceding one. Furthermore, owing to the variety of subject matter covered—chiral symmetry, functional techniques, current algebras—it is not possible to make this paper self-contained. In particular, derivations which are available in the literature have not been included. We hope to have remedied this defect by having included enough references.

II. FUNCTIONAL METHODS

In this section we define and state some properties of generating functionals commonly used in field theory and statistical mechanics. Several excellent reviews of these functional methods exist in the literature, to which we refer the reader who is not familiar with this subject.[5]

A. The functionals Γ and E

We will assume that we are working with an effective Hamiltonian[6]

$$H_{\text{eff}} = H + \sum_i \mu_i Q_i\,, \tag{2.1}$$

where the Q_i are conserved "charges" and the μ_i are the corresponding chemical potentials.[6] We assume that the fields are chosen such that $[Q_i, \varphi_\alpha] = q_{i\alpha}\varphi_\alpha$ and associate a chemical potential $\mu_\alpha = \sum_i q_{i\alpha}\mu_i$ with each field. The effective Lagrangian is then given by

$$\mathcal{L}_{\text{eff}}(\{\partial\varphi_\alpha\},\{\varphi_\alpha\}) = \mathcal{L}(\{(\partial_0 + i\mu_\alpha)\varphi_\alpha, \vec{\partial}\varphi_\alpha\},\{\varphi_\alpha\})\,. \tag{2.2}$$

The generating functional W for the connected Green's functions[5] is then defined by the functional integral

$$e^{-iW(J)} = \int [d\varphi]\exp\left[i\int \mathcal{L}_{\text{eff}} - i\int \sum_\beta J_\beta\varphi_\beta\right], \tag{2.3}$$

where the integration is over all fields which approach the ground-state (vacuum in field theory) expectation value $\langle\varphi_\alpha\rangle$ at infinity. The functional W generates all the connected Green's functions. For example, the ground-state expectation value is

$$\langle\varphi_\alpha(x)\rangle = \frac{\delta W}{\delta J_\alpha(x)}\bigg|_{J=0}\,, \tag{2.4}$$

and the propagator is

$$G_{\alpha\beta}(x,x') = \frac{\delta}{\delta J_\alpha(x)}\frac{\delta}{\delta J_\beta(x')}W\bigg|_{J=0}\,. \tag{2.5}$$

Finally, the ground-state energy per unit volume is $W(0)/\Omega$, where $\Omega = VT$ is the volume of space-time.

Using W we now define a c-number classical field by

$$\varphi_\alpha^{\text{cl}}(x) = \frac{\delta W}{\delta J_\alpha(x)}\,, \tag{2.6}$$

which may be interpreted as the expectation value $\langle\varphi_\alpha(x)\rangle_J$ in the presence of the sources J. Next a new functional[5] Γ is defined as the Legendre transformation

$$\Gamma(\varphi^{\text{cl}}) = \sum_\alpha \int \varphi_\alpha^{\text{cl}}(x)J_\alpha(x)dx - W\,, \tag{2.7}$$

where it is assumed that Γ has been expressed as a functional of φ^{cl} by using (2.6) to eliminate the J's. Since Γ is a Legendre transformation, we immediately deduce that

$$\frac{\delta\Gamma}{\delta\varphi_\alpha^{\text{cl}}(x)} = J_\alpha(x)\,; \tag{2.8}$$

similarly it follows that

$$\frac{\delta\Gamma}{\delta\varphi_\alpha^{\text{cl}}(x)}\bigg|_{\varphi_\alpha^{\text{cl}}(x) = \langle\varphi_\alpha(x)\rangle} = 0\,, \tag{2.9}$$

and that $-\Gamma(\langle\varphi\rangle)/\Omega = W(0)/\Omega$ is the ground-state energy per unit volume.

It can be shown that Γ is the generating functional for the *one-particle irreducible* Green's functions.[5] In particular, the inverse propagator is given by

$$G^{-1}{}_{\alpha\beta}(x, x') = \frac{\delta}{\delta\varphi^{\rm cl}_\alpha(x)} \frac{\delta}{\delta\varphi^{\rm cl}_\beta(x')} \Gamma \Big|_{\varphi^{\rm cl}=\langle\varphi\rangle} . \quad (2.10)$$

The three-point and higher vertices are obtained by taking higher derivatives of Γ around $\varphi^{\rm cl}=\langle\varphi\rangle$.

The rules for computing Γ in perturbation theory are the following.[5] Define

$$\overline{\mathfrak{L}}(\{\partial\varphi\}, \{\varphi\}) = \mathfrak{L}_{\rm eff}(\{\partial(\varphi+\varphi^{\rm cl})\}, \{\varphi+\varphi^{\rm cl}\}) - \mathfrak{L}_{\rm eff}(\{\partial\varphi^{\rm cl}\}, \{\varphi^{\rm cl}\}) \quad (2.11)$$

and make the decomposition

$$\overline{\mathfrak{L}} = \overline{\mathfrak{L}}_0 + \overline{\mathfrak{L}}_I + \text{terms linear in } \varphi , \quad (2.12)$$

where $\overline{\mathfrak{L}}_0$ contains all terms bilinear in φ and $\overline{\mathfrak{L}}_I$ contains the terms trilinear and higher. Any terms in $\overline{\mathfrak{L}}$ linear in φ can be ignored. This decomposition defines a set of Feynman rules, whose combinatorial and topological properties are the usual ones which can be read off from the functional integral in (2.3). The "free propagator" for the Feynman rules is

$$G_{\alpha\beta}{}^{-1}(x, x'; \{\varphi^{\rm cl}\}) = \frac{\delta}{\delta\varphi_\alpha(x)} \frac{\delta}{\delta\varphi_\beta(x')} \int \overline{\mathfrak{L}}_0 d^4x , \quad (2.13)$$

which depends on $\varphi^{\rm cl}$ through the dependence of $\overline{\mathfrak{L}}_0$ on $\varphi^{\rm cl}$, and the interaction vertices which also depend on $\varphi^{\rm cl}$ are contained in $\overline{\mathfrak{L}}_1$. In terms of the thus defined Feynman rules Γ is equal to the action of the classical field $\varphi^{\rm cl}$ plus the sum of all one-particle irreducible diagrams with no external lines. The diagrammatic expansion of Γ is conveniently organized into an expansion in the number of closed loops.[5] Through one closed loop Γ is

$$\Gamma = \int dx \mathfrak{L}_{\rm eff}(\{\partial\varphi^{\rm cl}_\alpha\}, \{\varphi^{\rm cl}_\alpha\}) - \frac{i\eta}{2} \operatorname{tr}\ln G_0(\varphi^{\rm cl}) + \text{two and more closed loops}, \quad (2.14)$$

$$\eta = \begin{cases} 1 \text{ for a boson loop} \\ -2 \text{ for a fermion loop} \end{cases} ,$$

where G_0 is given by (2.13) and the operation "tr ln" operates on the indices α, β as well as x and x'. The higher-order terms are straightforward Feynman diagrams built out of the propagator G_0 and the vertices $\overline{\mathfrak{L}}_I$.

The expansion for Γ is assumed to be fully renormalized.[7] Thus, $\mathfrak{L}_{\rm eff}$ in (2.14) contains only finite parameters. This implies, of course, that $\operatorname{tr}\ln G_0$ is to be regulated and appropriate subtractions are to be made.

In all cases of interest to us, $\varphi^{\rm cl}$ will be a function of the spatial coordinates $\vec{x}$ only. It is then useful to introduce a Fourier-transformed propagator $\tilde{G}$ by

$$G_{\alpha\beta}(x_0 - x_0', \vec{x}, \vec{x}'; \varphi^{\rm cl}) = \frac{1}{2\pi} \int d\omega\, \tilde{G}_{\alpha\beta}(\omega, \vec{x}, \vec{x}'; \varphi^{\rm cl}) \times e^{-i\omega(x_0 - x_0')} . \quad (2.15)$$

The Feynman diagrams can then be computed in ω space. The variable ω is conserved at the vertices generated by $\overline{\mathfrak{L}}_I$ and can be thought of as the effective energy (i.e., the true energy plus a chemical potential) associated with an internal line. The contour of the ω integrations in the diagrams is the same as in usual Feynman diagrams. It goes from $-\infty$ to 0 just below the real axis, crosses the real axis at $\omega = 0$, and proceeds to $+\infty$ just above the real axis.

When $\varphi^{\rm cl}$ is independent of time it is convenient to define the functional $E_{\rm eff}(\varphi^{\rm cl}) \equiv -\Gamma(\varphi^{\rm cl})/T$ and the classical potential energy

$$\mathcal{E}_{\rm cl}(\vec{x}) = -\mathfrak{L}_{\rm eff}(\{i\mu_\alpha\varphi^{\rm cl}_\alpha(\vec{x}),\ \vec{\partial}\varphi^{\rm cl}_\alpha(\vec{x})\},\ \{\varphi^{\rm cl}_\alpha(x)\}) . \quad (2.16)$$

Equation (2.14) then takes the form

$$E_{\rm eff}(\varphi^{\rm cl}) = \int \mathcal{E}_{\rm cl}(\vec{x}) d\vec{x} + \frac{i\eta}{4\pi} \int d\omega \operatorname{tr}\ln \tilde{G}_0(\omega; \varphi^{\rm cl}) + (\text{two and more closed loops}), \quad (2.17)$$

where the symbol "tr ln" now acts on the indices α, β and $\vec{x}$, $\vec{x}'$. Clearly $E_{\rm eff}$, like Γ, has an extremum at $\varphi^{\rm cl}=\langle\varphi\rangle$ and $E_{\rm eff}(\langle\varphi\rangle)$ is the ground-state (effective) energy. Finally, it is important to note that although $\varphi^{\rm cl}$ is independent of x^0 the time derivatives in $\mathfrak{L}$ do contribute to $\mathcal{E}_{\rm cl}$ because of the presence of the chemical potentials in the definition (2.2) of $\mathfrak{L}_{\rm eff}$.

B. Boson and fermion propagators

For bosons, the exact propagator $\tilde{G}(\omega; \langle\varphi\rangle)$ evaluated at $\varphi^{\rm cl}=\langle\varphi\rangle$ is known[8] to have its singularities along the real axis with the operator

$$\Delta(\omega) \equiv i[\tilde{G}(\omega + i\epsilon, \langle\varphi\rangle) - \tilde{G}(\omega - i\epsilon, \langle\varphi\rangle)]$$

being positive-definite for Re $\omega > 0$ and negative-definite for Re $\omega < 0$. If we take the volume to be finite so that the singularities are poles, this says that $\tilde{G}$ has poles of positive residue on the positive real axis and poles of negative residue on the negative real axis. Evidently, the contour of the ω integration is such that it goes above the positive-residue poles and below the negative-

residue poles. This is a consequence of the boundary conditions on the functional integral. It is always assumed that the approximate propagator $\tilde{G}_0(\omega, \varphi^{cl})$ has the analytic properties of $\tilde{G}(\omega, \langle\varphi\rangle)$. If it does not, ordinary perturbation theory for Γ will be meaningless. For a random choice of the arbitrary function φ^{cl}, $\tilde{G}_0$ may well fail to have the correct properties. What this would mean is that φ^{cl} is quite different from the true expectation value $\langle\varphi\rangle$. At a phase transition the expectation values $\langle\varphi\rangle$ change character. Thus the class of φ^{cl}'s which will give a propagator $\tilde{G}_0(\omega, \varphi^{cl})$ with the correct analytic properties also changes at a phase transition.[9]

As an example of this consider a free boson field carrying chemical potential μ. The inverse propagator in momentum space is then $(\omega + \mu)^2 - (m^2 + \vec{k}^2)$, so that the propagator has positive-residue poles at $\omega = |m^2 + k^2|^{1/2} - \mu$ and negative-residue poles at $\omega = -|m^2 + k^2|^{1/2} - \mu$. These poles are on the right sides of $\omega = 0$ only if $|\mu| \leq m$. This is the well-known fact that for a free boson the chemical potential cannot be greater than the mass. If one tries to make it bigger Bose-Einstein condensation occurs and $|\mu|$ remains at m.

For fermions, the analytic properties of the exact propagator $\tilde{G}(\omega, \langle\varphi\rangle)$ are similar (in that all the singularities lie on the real axis) but different in an important way. Because of the anticommutativity of Fermi fields $\Delta(\omega)$ is positive-definite for both positive and negative Re ω; i.e., all poles have positive residues. Because the residues do not have to change sign at Re $\omega = 0$, we never have the sort of trouble that leads to Bose-Einstein condensation. In fact, one can easily show that if φ^{cl} is real so that $\mathfrak{L}$ is Hermitian, then $\tilde{G}_0(\omega, \varphi^{cl})$ will always have the correct analytic properties.

C. Occupied fermion states

In cases of interest to us, φ^{cl} will be sufficiently close to the vacuum expectation value $\langle\varphi\rangle_{vac}$ such that the singularities of fermion propagators will retain something of their free-space structure. In particular, there will be a gap in the spectrum corresponding to the original vacuum gap between the fermion poles at positive energy and the antifermion poles at negative energy. The gap may, however, be shifted above or below $\omega = 0$ by a chemical potential which is somewhat larger than the fermion mass. How this will look is shown in Fig. 1.

When this remnant of the free-space gap exists, we can talk in a useful way about occupied fermion states. They are the poles of $\tilde{G}(\omega)$ which lie between the gap and the point Re $\omega = 0$ and correspond to those fermion levels which are occupied in the many-body ground state but would be empty in the vacuum. As an example of what this means, consider the one-loop term in (2.14). We write

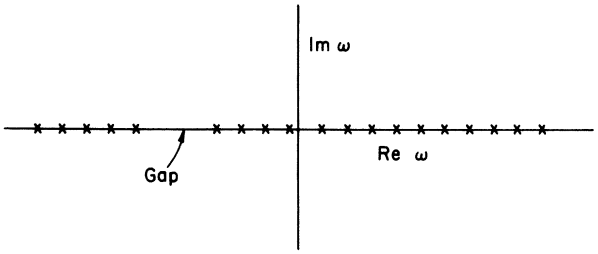

FIG. 1. The singularity structure of a fermion propagator assuming that $\varphi^{cl} - \langle\varphi\rangle_{vac}$ is not too large. The gap is the remnant of the usual gap between fermion and antifermion states.

$$\frac{1}{2\pi}\int_C d\omega\, \mathrm{tr}\ln\tilde{G}_0(\omega) = \frac{1}{2\pi}\int_{C_0} d\omega\, \mathrm{tr}\ln\tilde{G}_0(\omega) + \frac{1}{2\pi}\int_{C_1} d\omega\, \mathrm{tr}\ln\tilde{G}_0(\omega), \tag{2.18}$$

where C is the original contour shown in Fig. 2(a), C_0 is a contour that crosses the real ω axis in the gap, and C_1 is a loop around the occupied states. The integral along C_0 is divergent and must be renormalized. This may be accomplished by the standard methods[7] for renormalizing Γ. Turning to the integral along C_1, we integrate by parts to

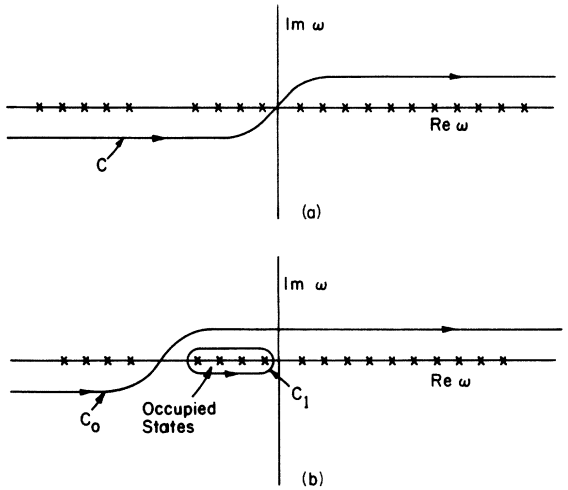

FIG. 2. The contour C shown in (a) is that which one is instructed to use in the perturbation expansion. In (b) it has been deformed into a contour C_0, which passes through the gap and a loop around the occupied states.

obtain

$$\frac{1}{2\pi}\int_{C_1} d\omega\, \mathrm{tr}\ln\tilde{G}_0(\omega) = \frac{1}{2\pi}\int_{C_1} \omega\, d\omega\, \mathrm{tr}[\gamma^0\tilde{G}_0(\omega)]$$
$$= i\sum_{\substack{\text{occupied}\\ \text{states}}} \epsilon_n\,, \qquad (2.19)$$

where ϵ_n is the effective energy of the nth state. Note that these effective energies, which include the chemical potentials, are always negative.

The above separation of an ω integration into contours C_0 and C_1 has a simple meaning in terms of Dirac hole theory. In hole theory the energy of a many-fermion system is the sum of the energies of all the fermion states which are filled in the many-body system minus the sum of the energies of the states which are filled in the vacuum. The occupied states inside the contour C_1 have no counterpart in the vacuum so we just add up their energies. The integral along C_0 contains the negative energy sea which we have to renormalize by subtracting the energy of the negative energy sea in the vacuum. After renormalization, this integral will be finite. It will also be nonzero unless $\varphi^{\mathrm{cl}} = \langle\varphi\rangle_{\mathrm{vac}}$.

For the general diagram in the expansion of Γ or E it is similarly useful to split integrations containing fermion propagators into pieces coming from the two contours C_0 and C_1. How this works is best explained by giving an example. Consider the typical two-loop diagram in Fig. 3, where the directed line in a fermion and the wiggly line is a meson. We break the diagram into three pieces: (a) where the ω integral is along C_0 for both fermion propagators, (b) where one integral is along C_0 and the other is along C_1, and (c) where both integrals are along C_1. The first piece (a) is highly divergent and ought to be lumped together with other such "vacuum" graphs and handled by renormalization methods.[10] Part (b) contains a divergent fermion self-energy which is removed by renormalization. After renormalization (b) is a finite self-energy summed over occupied states and represents a correction to the sum over lowest-order energies in (2.19). The third piece (c) can be recognized as a familiar term in the ground-state energy of a many-fermion system. Specifically, it is the (fermion) exchange part of the lowest-order energy coming from the meson exchange potential.[11] The same sort of decomposition can be made for any diagram in the expansion of Γ or E.

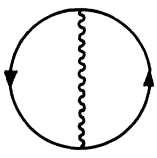

FIG. 3. A two-loop diagram. The directed line is the propagator of a fermion in the external field φ_{cl}. The wiggly line is a meson propagating in the same classical field. See the text for an explanation of the meaning of this diagram.

D. "Vacuum," "Hartree" and "correlation" energies

The essence of the previous subsection is that the functional $E_{\mathrm{eff}}(\varphi^{\mathrm{cl}})$ defined in the previous section can be conveniently separated into three pieces,

$$E_{\mathrm{eff}} = E^v + E^H + E^c\,, \qquad (2.20)$$

where the superscripts v, H, and c are chosen to suggest "vacuum," "Hartree," and "correlation" energies. They are defined as follows.[12]

The "vacuum" energy[13] is the sum of all diagrams in the perturbation expansion (2.17) for E_{eff}, with the integration contour taken along C_0 (Fig. 2) for all fermion propagators. The expansion for E^v starts in the tree approximation with

$$E^v(\varphi^{\mathrm{cl}}) = \int \mathcal{E}_{\mathrm{cl}}(\vec{x})d\vec{x} + (\text{closed loops})\,. \qquad (2.21)$$

The closed loops are divergent, but in a renormalizable theory there is an adjustable counterterm in $\mathcal{E}_{\mathrm{cl}}$ corresponding to each possible divergence. The computation and renormalization of E^v is a problem in relativistic field theory.

The "Hartree" energy E^H is the sum of all diagrams shown in Fig. 4, where the directed line is the fermion propagator *in the presence of the classical field* φ_{cl} *and the* ω *integration is around* C_1. The blobs in Fig. 4 are one-particle irreducible and if there are any internal fermion loops the integration is over C_0. The "Hartree" energy is explicitly given by

$$E^H(\varphi_{\mathrm{cl}}) = \sum_{\substack{\text{occupied}\\ \text{states}}} \epsilon_n'(\varphi_{\mathrm{cl}})\,, \qquad (2.22)$$

where the ϵ_n' are the positions of the poles of the

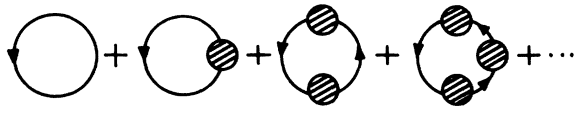

FIG. 4. The series of diagrams which sums to an energy shift for the occupied states. The blobs are one-particle irreducible and contain no sums over occupied states. The single directed lines are the propagator of a fermion in an external field; see the text.

dressed fermion propagator[14] obtained by summing the self-energy blobs shown in Fig. 4. When we minimize over φ_{cl} to get the ground-state energy the ϵ_n' are energies in a self-consistent field φ; hence the name Hartree. When there is more than one species of fermions present, the propagator is a (possibly diagonal) matrix and the ϵ_n' are the positions of the poles in the matrix or equivalently the zeros of the determinant of the inverse propagator. In lowest order E^H is

$$E^H(\varphi^{cl}) = \sum_{\substack{\text{occupied}\\ \text{states}}} \epsilon_n(\varphi^{cl}) + (\text{two and more closed loops}), \quad (2.23)$$

where the unprimed ϵ_n are the poles of the free fermion propagator in the presence of the classical field φ_{cl}.

The remaining diagrams which constitute E^c have at least two distinct loops, each of which contains a fermion propagator integrated around C_1. Part (c) of the diagram discussed at the end of Sec. (II) is an example of one contribution to E^c. It is evident that E^c is a many-body energy[15] with a self-consistent field piece taken out; hence the name "correlation."

III. PION CONDENSATION

In this section we discuss pion condensation in terms of the formal framework developed in Sec. II. Computational details which were amply illustrated in the preceding paper on the σ model will be omitted here.

A. The σ model

In the σ model we consider E_{eff} as a function of classical fields $\vec{\pi}^{cl}$ and σ^{cl}. The chiral-invariant

$$[\sigma^{cl}(\vec{x})]^2 + [\vec{\pi}^{cl}(\vec{x})]^2 = A^2 \quad (3.1)$$

is assumed to remain essentially equal to its vacuum value $A = \langle\sigma\rangle_{vac}$. The justification for imposing this constraint is that E^v will become very large if A differs significantly from $\langle\sigma\rangle_{vac}$. On the other hand, the chiral rotation of the vacuum fields

$$\begin{aligned} \sigma^{cl} &= \cos\theta\langle\sigma\rangle_{vac}, \\ \pi_1^{cl} &= \sin\theta\cos(\vec{k}\cdot\vec{x})\langle\sigma\rangle_{vac}, \\ \pi_2^{cl} &= \sin\theta\sin(\vec{k}\cdot\vec{x})\langle\sigma\rangle_{vac}, \\ \pi_3^{cl} &= 0 \end{aligned} \quad (3.2)$$

costs relatively little energy.[16] The particular form of the chiral rotation in (3.2) is that which is believed to correspond to charged pion condensation in a translationally invariant system.[1,2]

In the preceding paper we computed E^v for the fields in (3.2) using the simplest approximation of keeping only the term shown explicitly in (2.21). We found

$$\frac{E^v}{V} = (\langle\sigma\rangle_{vac})^2[\tfrac{1}{2}(\vec{k}^2 - \mu^2)\sin^2\theta + m_\pi^2(1-\cos\theta)], \quad (3.3)$$

where μ is the chemical potential for the third component of isospin and the symmetry breaking $C_1\sigma$ has been assumed. Although (3.3) was obtained from the "tree approximation," it can be shown that as a consequence of the chiral symmetry in the model *Eq. (3.3) is actually exact up to small terms of order* m_π^4, k^4, *and* μ^4. This will be discussed in more detail below. We are justified in considering k^4 and μ^4 as small since we expect condensation to occur for values of k and μ of order m_π. Note that E^v as given by (3.3) remains of order $(\langle\sigma\rangle_{vac})^2 m_\pi^2$ for all values of the rotation angle θ. In contrast, a fractional change in A of order unity would lead to a large E^v of order $\langle\sigma_{vac}\rangle^2 m_\sigma^2 \gg \langle\sigma_{vac}\rangle^2 m_\pi^2$.

In our model calculations we approximated the "Hartree" energy of the nucleons by the first term in Eq. (2.23). This approximation gives the terms of order $|k|$ and $|\mu|$ in E^H correctly but misses terms of order k^2, μ^2, and m_π^2. The latter terms, which are likely to be important in actual neutron-star calculations, will be discussed below.

The "correlation" energy E^c was completely ignored in the model calculations. It cannot be ignored in a quantitative calculation and its omission is a major fault of the models.[17]

Our final remark about the σ model is that according to the formalism of Sec. II one should use renormalized parameters in E^v and E^H. Thus g_A is not constrained to its bare value $g_A = 1$. After renormalization, the only constraint is that the Goldberger-Treiman relation be respected.

B. A model-independent approach

The σ model is not to be taken literally as a correct model of π-N and N-N interactions. Its virtue is as an example of the consequences of chiral symmetry. Here we will look at pion condensation in a model-independent way.

We will assume that the strong interactions can be described by a Lagrangian of the form

$$\mathcal{L} = \mathcal{L}_{sym} + \mathcal{L}_{SB}, \quad (3.4)$$

where the symmetrical Lagrangian $\mathcal{L}_{sym}$ is invariant under chiral[3] SU(2)⊗SU(2) and the symmetry-breaking piece $\mathcal{L}_{SB}$ is a small perturbation.

The perturbation $\mathcal{L}_{SB}$ conserves isospin and will be assumed to contain no derivatives of the fields.

Forgetting the many-body problem for the moment, we can imagine taking this Lagrangian with all chemical potentials equal to zero and minimizing Γ to find the vacuum values $\langle\varphi_i\rangle_{vac}$ of various fields φ_i. Chiral symmetry is supposed to be spontaneously broken, which means that some fields which are not chiral singlets have vacuum values which remain finite as $\mathcal{L}_{SB} \to 0$. A simple example of such a field is the σ field in the σ model.

In the σ model we argued that pion condensation is quite well described by fields π^{cl} and σ^{cl} which are a chiral rotation of the vacuum fields. We now assert that the situation will be similar in the general case. The reason is the same as before. When θ is not small, our rotation leads to order one changes in the fields. However, since the theory is nearly chirally symmetric this does not lead to a large E^v. Any other kind of order unity shift away from the vacuum such as a change in a "radius" like $(\sigma^2 + \vec{\pi}^2)^{1/2}$ would make E^v very large. Of course, to get quantitative answers we should allow for small changes in chiral invariants. Also, we have to remember that some fields which cannot have vacuum expectation values will acquire expectation values in a many-body system. An example of such a field would be the time component of a four-vector. Our assertion is basically that any changes in chiral invariants and the additional many-body expectation values are qualitatively unimportant compared to the rotation of the vacuum fields.

The basic hypothesis as described above makes sense only if θ is not too small. For small θ the rotation is small and is not easily distinguishable from many other small effects. Assuming the phase transition to be second order, θ begins to move away from zero at some critical density and presumably increases as the density increases beyond the critical point. The number of condensed pions is roughly proportional to $\sin^2\theta$, which starts at zero and increases as θ heads to some angle of reasonable magnitude where one could say that the condensation is more or less fully developed. *It is in this fully developed condensation that chiral symmetry plays a particularly important role.* Of course, it is also interesting to look at small θ, in particular to find the critical density at which the phase transition occurs. For small θ, there is nothing wrong with our general formalism or use of chiral symmetry. The point is that for small θ there is no good reason to expect that our methods are better than any other schemes.

We assume, therefore, that for a fully developed condensation, the appropriate fields to insert in E_{eff} are not too different from chirally rotated vacuum fields. The particular rotation which we choose is the abstract chiral rotation corresponding to rotation of the σ and π fields shown in Eq. (3.2). Actually, rather than work with the rotated fields it is easier to apply the counterrotation to the Lagrangian and work around the ordinary vacuum fields at the expense of having to deal with a modified Lagrangian. If the rotation in (3.2) did not depend on $\vec{x}$, the symmetric piece of the Lagrangian, $\mathcal{L}_{sym}$, would not be changed by the counterrotation. It does depend on $\vec{x}$, but the way in which a symmetric Lagrangian transforms under a local rotation is well known. Since by assumption the symmetry-breaking term $\mathcal{L}_{SB}$ conserves isospin and contains no derivatives of the fields, the inverse of the rotation in (3.2) applied to $\mathcal{L}_{SB}$ will have the same effect as the global rotation obtained by setting $\vec{k}=0$ and replacing θ by $-\theta$ in (3.2). To say much more about the result of rotating $\mathcal{L}_{SB}$ requires a model of the sort to be discussed below.

We now quote the result of applying the inverse of the rotation in (3.2). In doing so, it is convenient to define a four-vector

$$k^v = (\mu, k_1, k_2, k_3) \tag{3.5}$$

built out of the chemical potential μ for the third component of isospin and the three-vector $\vec{k}$. We also introduce the vector (isospin) currents V_i^v, $i = 1, 2, 3$ and the axial-vector currents A_i^v, $i = 1, 2, 3$, whose charges generate $SU(2)\otimes SU(2)$. The rotation changes the effective Lagrangian, containing chemical potentials, according to

$$\mathcal{L}_{eff} \to \mathcal{L}_{sym} - \cos\theta\, k_v V_3^v - \sin\theta\, k_v A_2^v + k^v k^\lambda S^{v\lambda}(\theta) + \mathcal{L}_{SB}(\theta), \tag{3.6}$$

where $\mathcal{L}_{SB}(\theta)$ is the rotated symmetry-breaking term and $S^{v\lambda}(\theta)$ comes from the terms in the Lagrangian which are bilinear in derivatives of the fields. We do not have to worry about the detailed structure of $S^{v\lambda}$. In diagrammatic language $S^{v\lambda}$ is responsible for the "seagull" diagrams which will appear in matrix elements involving two or more of the currents $k_v V_3^v$ or $k_v A_2^v$. The structure of these seagulls is known to be completely determined by the Ward identities of the theory.[18] In practice the best thing to do is to compute the physical quantities using the covariant Feynman rules specified by the functional integral. The seagull diagrams will then take care of themselves and the Ward identities will be satisfied.

To get a feeling for what $\mathcal{L}_{SB}(\theta)$ looks like let us consider the popular model where $\mathcal{L}_{SB}$ belongs to the $(\frac{1}{2}, \frac{1}{2})$ representation of $SU(2)\otimes SU(2)$. The

$(\frac{1}{2}, \frac{1}{2})$ representation contains fields u and v_1, v_2, v_3 with exactly the same quantum numbers and transformation properties as the fields σ and π_1, π_2, π_3 in the σ model. The symmetry-breaking Lagrangian is taken to be the parity-even isospin-zero field u and $\mathcal{L}_{SB}(\theta)$ is then

$$\mathcal{L}_{SB}(\theta) = \cos\theta\, u - \sin\theta\, v_1 . \tag{3.7}$$

At present there is little experimental support for this particular $\mathcal{L}_{SB}$. A good determination of the π-π scattering lengths would be decisive.[19]

There is some model-independent information available about the operator

$$\Sigma = \frac{d^2}{d^2\theta}\, \mathcal{L}_{SB}(\theta)\Big|_{\theta=0} , \tag{3.8}$$

which is just minus u in the $(\frac{1}{2}, \frac{1}{2})$ model. The vacuum expectation value is known to be[20]

$$\langle 0|\Sigma|0\rangle = -f_\pi{}^2 m_\pi{}^2 + O((\mathcal{L}_{SB})^2) , \tag{3.9}$$

where we have indicated that (3.9) is correct up to second order in $\mathcal{L}_{SB}$ and introduced the constant f_π defined by[21]

$$\langle \pi_\alpha(q)|A^\nu_\beta(0)|0\rangle = -iq^\nu \delta_{\alpha\beta} f_\pi , \tag{3.10}$$

with a pion state normalized to $\langle \vec{q}|\vec{q}'\rangle = 2q^0\delta^3(\vec{q}-\vec{q}')$. The matrix element of Σ between nucleons can be determined from an analysis of on-shell π-N scattering. It appears that[22,23]

$$\langle N|\Sigma|N\rangle = 50\text{–}75\ \text{MeV} + O((\mathcal{L}_{SB})^2) , \tag{3.11}$$

where the sign is definite but the magnitude is somewhat uncertain. The normalization in (3.11) is such that in the $(\frac{1}{2}, \frac{1}{2})$ model where $\Sigma = -\mathcal{L}_{SB}$, the symmetry breaking increases the nucleon mass by the amount on the right-hand side of the equation.

The only other new terms appearing in (3.6) are the vector and axial-vector currents V_3^ν and A_2^ν, whose properties are well known.

Our task is now to compute E_{eff} for the Lagrangian (3.6) under the assumption that the fields do not differ very much from the vacuum fields. Assuming that the vacuum fields are known, this is equivalent to computing the corrections to the (effective) ground-state energy $E_{\text{eff}}(\theta)$ due to the extra terms in the Lagrangian (3.6). Breaking this energy into pieces E^v, E^H, and E^c we treat each piece separately.

C. The "vacuum" energy

It can be shown that up to order $(\mathcal{L}_{SB})^2$, k^4, and μ^4 the "vacuum" energy is exactly equal to

$$\frac{E^v}{V} = f_\pi{}^2[\tfrac{1}{2}(\vec{k}^2 - \mu^2)\sin^2\theta] + \langle 0|\mathcal{L}_{SB}(0) - \mathcal{L}_{SB}(\theta)|0\rangle + O(\mathcal{L}_{SB}{}^2, \vec{k}^4, \mu^4) . \tag{3.12}$$

Using (3.9), it is easily seen that in the $(\frac{1}{2}, \frac{1}{2})$ symmetry-breaking model (3.12) reduces to

$$\frac{E^v}{V} = f_\pi{}^2[\tfrac{1}{2}(\vec{k}^2 - \mu^2)\sin^2\theta + m_\pi{}^2(1-\cos\theta)] + O(\mathcal{L}_{SB}{}^2, \vec{k}^4, \mu^4) . \tag{3.13}$$

For any type of symmetry breaking, it follows from (3.9) that for small θ

$$\frac{E^v}{V} = f_\pi{}^2(\vec{k}^2 - \mu^2 + m_\pi{}^2)\frac{\theta^2}{2} + O(\theta^4) + O(\mathcal{L}_{SB}{}^2, \vec{k}^4, \mu^4) . \tag{3.14}$$

Equation (3.12) can be derived in various ways. It was obtained by a straightforward operator approach in Ref. 4. A more elegant approach is to realize that the way in which the vacuum functional Γ transforms under local chiral rotations has been exhaustively studied using the nonlinear representation method.[24] This method also yields the formula in (3.12) and contains an error of the indicated order.

Our formula for E^v illustrates some of the power of the chiral approach to pion condensation. In physical terms, E^v contains all the effects of π-π scattering, the energy associated with irreducible six-pion vertices and so on. Up to the indicated order in $\vec{k}$, μ, and $\mathcal{L}_{SB}$, it also includes the difference in energy between the negative-energy sea of antinucleons in the presence of a pion condensate and the negative-energy sea in the vacuum. All such effects are lumped into one simple formula. The physical pion mass and (3.9) set the scale of $\langle 0|\mathcal{L}_{SB}|0\rangle$. There are no divergences in (3.12); only physical renormalized parameters appear.

D. The "Hartree" energy

The candidates for fermions in a neutron star are nucleons, hyperons, and the (3, 3) resonance Δ. We will treat only nucleons here. The extension to hyperons is straightforward and the inclusion of Δ's was discussed in the preceding paper.

To obtain the energies of the occupied states we need poles of the nucleon propagator. In general it is impossible to define the nucleon propagator in a model-independent way. However, in a neutron star we expect that off-mass-shell effects will be small. The nucleon propagator will then be the free propagator $(i\not{p} - M)^{-1}$ modified by the extra terms which appear in the rotated Lagrangian. The simplest approximation is to

keep only the vector and axial-vector currents in (3.6) which yields for the inverse propagator

$$S^{-1} = i\not{\partial} - M - \cos\theta\,\frac{\tau_3}{2}\not{k} - g_A \sin\theta\,\frac{\tau_2}{2}\,\gamma_5\not{k} + \gamma^0\mu_B + O(k^2, \mathcal{L}_{SB}), \tag{3.15}$$

where μ_B is the chemical potential for baryon number and the indicated error of order $\mathcal{L}_{SB} \sim m_\pi^2 \sim k^2$ comes from neglecting $\mathcal{L}_{SB}$, $k^\mu k^\lambda S_{\mu\lambda}$, and the second-order contribution of $k_\mu A_2^\mu$. The approximation in (3.15) was used in our model calculations.

One can do better. Consider first the term $\mathcal{L}_{SB}(\theta)$. It is convenient to break it into even and odd pieces,

$$\begin{aligned}\mathcal{L}_{SB}(\theta) &= \tfrac{1}{2}[\mathcal{L}_{SB}(\theta) + \mathcal{L}_{SB}(-\theta)] \\ &\quad + \tfrac{1}{2}[\mathcal{L}_{SB}(\theta) - \mathcal{L}_{SB}(-\theta)] \\ &= \mathcal{L}^e_{SB}(\theta) + \mathcal{L}^o_{SB}(\theta),\end{aligned} \tag{3.16}$$

which have, respectively, even and odd parity. The odd-parity piece will have small matrix elements between nonrelativistic nucleon states. Since nucleons are expected to be reasonably nonrelativistic in neutron stars and $\mathcal{L}_{SB}$ is already a small operator we ignore this piece. For the even part, we define

$$\Delta M(\theta) \equiv -\langle N|\,\mathcal{L}^e_{SB}(\theta) - \mathcal{L}_{SB}(0)\,|N\rangle\,, \tag{3.17}$$

where the normalization is the same as in (3.11) and we need not worry about isospin since $\mathcal{L}^e_{SB}(0)$ contains only even isospin and only its $I=0$ component survives in (3.17). The first-order effect of $\mathcal{L}_{SB}$ on S^{-1} will be to replace M in (3.15) by $M + \Delta M(\theta)$. From (3.11) we see that ΔM decreases for small θ. This means that the nucleon mass is going down, which will encourage condensation. For finite θ one cannot make a general statement, but in the $(\frac{1}{2}, \frac{1}{2})$ model $\Delta M(\theta)$ is easily seen to be a positive number (of order 50 to 75 MeV) times $(\cos\theta - 1)$, which is negative for all $\theta \neq 0$.

Next we need the second-order contribution of $k_\nu A_2^\nu$. This must be computed along with the first-order term in $k_\nu k_\lambda S^{\nu\lambda}$. Together they will add a term to the inverse propagator

$$-\frac{i}{2}\int d^4x\langle N|\,T'(k_\nu A_2^\nu(x) k_\lambda A_2^\lambda(0))\,|N\rangle\,\sin^2\theta - \langle N|\,k_\nu k_\lambda S^{\nu\lambda}(\theta)\,|N\rangle$$
$$= -\frac{i}{2}\int d^4x\langle N|\,T'^*(k_\nu A_2^\nu(x) k_\lambda A_2^\lambda(0))\,|N\rangle\,\sin^2\theta \equiv \tfrac{1}{2}\sin^2\theta\, k_\nu k_\lambda \alpha^{\nu\lambda}, \tag{3.18}$$

where the last line defines $\alpha^{\mu\nu}$. In (3.18) we have introduced the covariant time-ordered product T^* and the prime on T and T^* means that, since we are computing the inverse propagator, the single-nucleon intermediate state should be omitted. That the ordinary time-ordered product T combines with $S^{\nu\mu}$ to give a single covariant time-ordered product T^* is a consequence of the relation between seagull diagrams and Ward identities discussed above.[18]

The quantity $k_\nu k_\lambda \alpha^{\nu\lambda}$ can be related to an on-shell π-N scattering amplitude as follows. In the scattering amplitude for $\pi_1(k) + N(p) \to \pi_1(k') + N(p')$, where $k^2 = k'^2 = m_\pi^2$ and $p^2 = p'^2 = M^2$, we take as independent variables $k\cdot k' = m_\pi^2 - \frac{1}{2}t$ and $\nu = (p+p')\cdot(k+k')/4M = (s-u)/4M$. In terms of the usual isospin-even amplitudes[23,25] $A^{(+)}$ and $B^{(+)}$, we define $D = (A^{(+)} + \nu B^{(+)}$ minus the pseudovector Born term). Explicitly, D is given in the standard notation by

$$D = A^{(+)} + \nu B^{(+)} - \frac{G^2}{2M}\left(\frac{\nu_B}{\nu_B + \nu} + \frac{\nu_B}{\nu_B - \nu}\right), \quad \nu_B = -\frac{k\cdot k'}{2M} \tag{3.19}$$

where G is the π-N coupling constant ($G^2/4\pi \approx 14$).

It is a standard result of partial conservation of axial-vector current (PCAC) and current algebra that[22,26]

$$f_\pi^2 D(\nu, k\cdot k') = \alpha^{\mu\lambda} k^\mu k'^\lambda + \langle N|\,\Sigma\,|N\rangle + O((\mathcal{L}_{SB})^2, (k\cdot k')^2, \nu^2)\,. \tag{3.20}$$

If we define the expansion

$$D(\nu, k\cdot k') = D(0,0) + d_1\nu^2 + d_2(k\cdot k') + \cdots, \tag{3.21}$$

where we have used the fact that D is an even function of ν, then

$$\alpha^{\mu\lambda} k^\mu k^\lambda = f_\pi^2(d_1\nu^2 + d_2 k^2) + O((\mathcal{L}_{SB})^2)\,. \tag{3.22}$$

The numbers d_1 and d_2 can be determined experimentally from the on-shell π-N dispersion relations.[23,27] [In using experimental data the reader is reminded that the momentum transfer t is $(2m_\pi^2 - 2k\cdot k')$, so that setting $k = k'$ as in (3.22) does not correspond to an on-shell forward scattering unless $k^2 = m_\pi^2$.]

To use (3.22) in the nucleon propagator, it is convenient to assume nonrelativistic nucleons so that $\nu \approx k^0 = \mu$.

Putting together (3.17), (3.18), and (3.22) yields an inverse propagator

$$S^{-1} = i\partial\!\!\!/ - M - \Delta M(\theta) - \cos\theta\,\frac{\tau_3}{2}\,\mu\!\!\!/ - g_A \sin\theta\,\frac{\tau_2}{2}\,\gamma_5 k\!\!\!/ + \frac{f_\pi^2 \sin^2\theta}{2}\,(d_1\mu^2 + d_2 k_\lambda k^\lambda) + \gamma^0\mu_B + O((\mathfrak{L}_{SB})^2, \vec{k}^4, \mu^4)\,,$$
$$k^\lambda = (\mu, k_1, k_2, k_3), \quad (3.23)$$

where we remind the reader that μ is the chemical potential for the third component of isospin, μ_B is the chemical potential for baryon number, and in addition to the indicated error we have made the nonrelativistic approximations of setting $\nu = \mu$ and dropping $\mathfrak{L}_{SB}{}^2(\theta)$.

With this propagator one can evaluate E^H to the same order of approximation as was attained for E^v. Since the new terms in (3.23) do not contain derivatives or γ or τ matrices they simply make an additive correction to the energies of the occupied states ϵ_n. Consequently, one finds, setting $k_\lambda k^\lambda = \mu^2 - \vec{k}^2$,

$$E^H = E_0^H + N_B\left\{\Delta M(\theta^2) - \frac{f_\pi^2 \sin^2\theta}{2}\left[(d_1 + d_2)\mu^2 - d_2\vec{k}^2\right]\right\}$$
$$+ O((\mathfrak{L}_{SB})^2, \mu^4, \vec{k}^4)\,, \quad (3.24)$$

where N_B is the number of baryons and E_0^H is the "Hartree" energy computed with the simpler propagator (3.15). In the preceding paper E_0^H was evaluated for nonrelativistic nucleons.

Again we see that chiral symmetry gives a simple general result that depends only on experimentally determined parameters. To the indicated order of approximation, (3.24) for E^H takes account of the fact that the condensed pions are off the mass shell. It also takes into account direct interactions of a nucleon with two and more pions; i.e., processes which in nuclear physics language would be called true multibody forces. [Note that since $\Delta M(\theta)$ is not in general proportional to $\sin^2\theta$, the quantity multiplying N_B in (3.24) is not $\sin^2\theta$ times a π-N scattering amplitude.] However, the formula for E^H may not be as useful as the corresponding expression for E^v. The reason is that for values of $\vec{k}$ and μ of interest in neutron stars the expansion in Eq. (3.21) may not be useful. This will be discussed in Sec. V.

E. The "correlation" energy

It is not clear that our chiral rotation approach to pion condensation has much that is useful to say about E^c. The reason is that in E^v and E^H (as we have defined them) the basic scale of energy is set by high-energy physics and is, say, a nucleon mass. For E^v and E^H, therefore, it is useful to think of μ^2, $\vec{k}^2$, and $m_\pi{}^2$ as small parameters. For the correlation energy E^c, the basic scale is set by a nuclear many-body problem. On this scale μ^2, $\vec{k}^2$, and $m_\pi{}^2$ are large.

We can remark that one is only interested in the way that E_c depends on the θ-dependent terms in the rotated Lagrangian in (3.6). It depends on θ for two reasons: First because the energies and wave functions of the occupied states depend on θ and second through an explicit modification of the nucleon-nucleon potential. Since the nuclear force is supposed to be basically chirally symmetric, one expects that the latter effect is very small except for the special case of the pion exchange potential. The pion propagator will be substantially modified by the rotation and any attempt to compute E^c should take this into account.

To our usual approximation the pion propagator is completely determined by chiral-symmetry considerations. Like E^v, it is correctly given by the tree diagrams in the σ model, provided that one chooses the correct symmetry-breaking term and uses physical values for $m_\pi{}^2$ and f_π.

IV. BROKEN SYMMETRY

We started with a Lagrangian that is translationally invariant and is invariant under rotations around the third axis in isospin space. The fields in (3.2) violate both of these symmetries. We note, however, that a translation $\vec{x} \to \vec{x} + \vec{a}$ combined with an I_3 rotation through an angle $\vec{k}\cdot\vec{a}$ does not affect the field. In any case, we have a spontaneously broken symmetry. If there are no long-range forces the Goldstone theorem tells us to expect a boson mode whose energy vanishes as its wavelength goes to infinity. In the real world electromagnetism provides a long-range force, and since the Goldstone mode is charged in this case, one expects that it will turn into a plasma oscillation. In the preceding paper we showed that in the absence of nucleons this is precisely what happens. In fact, the result is general as we shall show below.

The fields in (3.2) also break rotational invariance. One might worry that there is another Goldstone boson associated with the breakdown of this symmetry. It could not be found in a model with mesons alone since in that case condensation occurs in a $\vec{k} = 0$ state. We will show below that there is not in fact a second Goldstone boson. All the broken symmetries are taken care of by a single boson which becomes a plasmon upon

interaction with the photon. As asserted in the preceding paper, there is then no reason to expect additional low-lying boson modes.

A. Electromagnetism and the breakdown of I_3

For definiteness we will discuss the question in terms of the σ model. The conclusions, however, will be perfectly general. In the condensed phase, we expect that E_{eff} will be minimized for classical fields

$$\begin{aligned} \sigma^{\text{cl}} &= a\cos\theta, \\ \pi_1^{\text{cl}} &= a\sin\theta\cos(\vec{k}\cdot\vec{x}), \\ \pi_2^{\text{cl}} &= a\sin\theta\sin(\vec{k}\cdot\vec{x}), \\ \pi_3^{\text{cl}} &= 0, \end{aligned} \tag{4.1}$$

where $a \approx \langle\sigma\rangle_{\text{vac}}$. To find the excitations, we then need to evaluate the functional Γ for classical fields which are infinitesimally close to the ground-state fields in (4.1). The methods of Sec. II will then tell us how to compute propagators, vertices, and so on. We know that the Goldstone particle comes from π_1 and π_2 so let us concentrate on them. A possible parametrization of these fields is

$$\begin{aligned} \pi_1^{\text{cl}}(\vec{x}, t) &= [1+\rho(\vec{x}, t)]\, a\sin\theta\cos(\vec{k}\cdot\vec{x} + \varphi(\vec{x}, t)), \\ \pi_2^{\text{cl}}(\vec{x}, t) &= [1+\rho(\vec{x}, t)]\, a\sin\theta\sin(\vec{k}\cdot\vec{x} + \varphi(\vec{x}, t)). \end{aligned} \tag{4.2}$$

This decomposition of π_1 and π_2 into a "radius" and an angle φ is nonsingular for the small values of φ and ρ of interest.

Consider now Γ as a functional of $\varphi(\vec{x}, t)$ with $\rho = 0$ and all other fields set equal to their ground-state expectation values. As long as φ has nontrivial space-time dependence $\Gamma(\varphi)$ will be greater than $\Gamma(0)$. However, in the limit that the space-time dependence of φ vanishes, $\Gamma(\varphi)$ must approach $\Gamma(0)$. This must be so because for constant φ and $\rho = 0$, (4.2) is just a rotation $\exp(i\varphi I_3)$ on the ground state. This cannot change the energy because H_{eff} commutes with I_3. This is the origin of the Goldstone boson. Its "field" is $\varphi(\vec{x}, t)$ and it takes no energy to create it in the limit of infinite wavelength. To make contact with the model calculation of the preceding paper, we expand around the ground-state fields and find to lowest order

$$a\sin\theta\,\varphi(\vec{x}, t) = \pi_2^{\text{cl}}(\vec{x}, x)\cos(\vec{k}\cdot\vec{x}) - \pi_1^{\text{cl}}(\vec{x}, t)\sin(k\cdot\vec{x}) + \cdots . \tag{4.3}$$

For mesons alone condensation occurs for $\vec{k} = 0$ and $a\sin\theta\,\varphi$ becomes the Goldstone boson field π_2 of the model calculation.

When the electromagnetic field is included φ can be eliminated from Γ by the gauge transformation[28]

$$A^\mu \to A^\mu + \frac{1}{e}\,\partial^\mu\varphi \ . \tag{4.4}$$

All the low-lying modes are then basically electromagnetic. To compute these modes would thus require a detailed consideration of the effects of electrons, etc.

B. Rotational invariance

We now turn to rotational invariance. Because we already understand its effect, electromagnetism will be ignored.

We will work in the chirally rotated frame where the expectation values are normal but the Lagrangian is that given in (3.6). In going to this frame the conserved current V_3^μ rotates by

$$V_3^\mu \to J^\mu \equiv \cos\theta\, V_3^\mu + \sin\theta A_2^\mu \ . \tag{4.5}$$

The current J^μ is of course conserved when we use the rotated Lagrangian (3.6). The ground state is, however, not an eigenstate of the conserved charge $\int d^3x J^0$. This is another way to see that we need the Goldstone boson. The Goldstone particle is produced by the current acting on the ground state.

To see what happens with rotational invariance, we want to chirally rotate the momentum density T^{0i}. To do this we need the equal-time commutator

$$[T^{0i}(\vec{x}, t), K^0(\vec{y}, t)] = i\,\frac{\partial}{\partial y^i}(\delta^3(\vec{x}-\vec{y})K^0(\vec{y}, t)), \tag{4.6}$$

where K^0 is the time component of any one of the vector or axial-vector currents. One then easily calculates that under the rotation

$$T^{0i} \to T^{0i} + k^i J^0 . \tag{4.7}$$

To check the conservation law we use the fact that

$$H_{\text{eff}} \to H_{\text{eff}} - \vec{k}\cdot\int \vec{J}\, d^3x + \text{"Schwinger terms"} , \tag{4.8}$$

where the Schwinger terms can be ignored in the following calculation. Then using the commutator

$$[T^{0i}(\vec{x}, t), J^j(\vec{y}, t)] = i\,\frac{\partial}{\partial y^i}(\delta^3(\vec{x}-\vec{y})J^j(\vec{y}, t)) - i\,\frac{\partial}{\partial x^j}(\delta^3(\vec{x}-\vec{y})J^i(\vec{x}, t)) , \tag{4.9}$$

one finds

$$i\left[H_{\text{eff}} - \vec{k}\cdot\int \vec{J}\, d^3x,\ T^{0i}(\vec{x},t) + k^i J^0(\vec{x},t)\right]$$
$$= -\frac{\partial}{\partial x^j}\left(T^{ij}(\vec{x},t) + k^i J^j(\vec{x},t) + k^j J^i(\vec{x},t)\right), \tag{4.10}$$

where the right-hand side is the divergence of a symmetric tensor, as it should be.

Equation (4.10) is the local version of momentum and angular momentum conservation in our chirally rotated frame. The ground state is not an eigenstate of the charges associated with $T^{0i} + k^i J^0$. Therefore, there must be a Goldstone boson. But this boson is again made by acting with J^0 on the ground state and is therefore the same one as before. In both the I_3 current and the momentum current, the same Goldstone particle arises from the same operator J^μ. We note that the breaking of rotational invariance here is of a very special type. For this special type of symmetry breaking, there is not an extra Goldstone boson.

V. REALISTIC π-N INTERACTIONS

In Sec. III we saw that chiral symmetry instructs us to compute the (in general) off-shell π-N scattering amplitude by setting it equal to the measurable on-shell amplitude at the same values of ν and $k\cdot k'$. For the pion-condensation problem at least, the error incurred in this extrapolation is of second order in the amount by which the pions are off-shell.

This section is devoted to some remarks about the observed π-N amplitudes as extrapolated in this way. We will assume nonrelativistic nucleons, so that the approximation

$$k^0 \approx k'^0 \approx \nu \approx \mu \tag{5.1}$$

is valid. For pion condensation, we need only the combinations $A^{(x)} + \nu B^{(x)}$, where A and B are the standard amplitudes[23] and the index x can stand for a definite isospin $I = \frac{1}{2}$ or $\frac{3}{2}$ or a definite crossing + or − .

A. Chiral-symmetry predictions

Chiral symmetry says that

$$[A^{(+)} + \nu B^{(+)} - (\text{pseudovector Born term})]\big|_{\nu=0;\, k\cdot k'=0}$$
$$\equiv D\big|_{\nu=0;\, k\cdot k'=0} = (f_\pi)^{-2}\langle N|\Sigma|N\rangle\ , \tag{5.2}$$

which was already used in the previous section. It also predicts that

$$A^{(-)} + \nu B^{(-)} = \frac{G^2}{M}\frac{\nu\nu_B}{\nu_B^2 - \nu^2} - \frac{G^2\nu}{2M^2} + \tfrac{1}{2}(f_\pi)^{-2}\nu + O(\nu^3, \nu k\cdot k'), \tag{5.3}$$

where the (−) Born term is given in the same notation as in Eq. (3.19). The result for the symmetric amplitude in Eq. (5.2) tells us how to measure $\langle N|\Sigma|N\rangle$. For the antisymmetric (−) amplitude one has a prediction which has been well confirmed experimentally. The term $(f_\pi)^{-2}\nu$ appears as part of $[\cos(\theta\tau_3/2)\not{k}]$ in our expression for the nucleon propagator.

B. The symmetric amplitude below threshold

The coefficients in the expansion in Eq. (3.22) have been extensively studied through dispersion relations.[23,27] The various numbers have not yet been completely fixed in a quantitative way but a clear qualitative picture is available. The amplitude at threshold

$$D(\text{threshold}) \cong D(0,0) + m_\pi^2(d_1 + d_2) \tag{5.4}$$

is very small. Numerically, it is something like a factor of 10 smaller than $D(0,0)$. This comes about because of a presumably accidental cancellation between the two terms on the right-hand side of (5.4) and is responsible for the well-known fact that the isospin-even scattering length is very small.[23] The coefficients d_1 and d_2 are comparable in magnitude so that this peculiar cancellation comes from both the ν^2 and $k\cdot k'$ dependence.

C. The partial-wave expansion

Turning to the expansion in partial waves, we assume nonrelativistic nucleons, drop small terms of order m_π/M, and keep only S and P waves. The expansion is then

$$A^{(x)}(\mu, \vec{k}\cdot\vec{k}') + \mu B^{(x)}(\mu, \vec{k}\cdot\vec{k}')$$
$$= 4\pi\left\{g_0^{(x)}(\mu) + \left[2g_{1+}^{(x)}(\mu) + g_{1-}^{(x)}(\mu)\right]\vec{k}\cdot\vec{k}'\right\}, \tag{5.5}$$

where we set $\nu = k^0 = k^{0\prime} \equiv \mu$ so that $k\cdot k' = \mu^2 - \vec{k}\cdot\vec{k}'$ and $g_{L\pm}$ is a partial-wave amplitude for $J = L \pm \frac{1}{2}$. The normalization is such that for amplitudes of definite isospin $I = \frac{1}{2}$ or $\frac{3}{2}$

$$g_0^{(I)} = \frac{e^{i\delta}\sin\delta}{q}\ ,$$
$$g_{1\pm}^{(I)} = \frac{e^{i\delta}\sin\delta}{q^3}\ , \tag{5.6}$$
$$q = (\mu^2 - m_\pi^2)^{1/2}\ ,$$

where the phase shifts are to be evaluated at center-of-mass energy $\sqrt{s} = M + \mu$. Thus, we see that for S- and P-wave scattering off nonrelativistic nucleons, the chiral prescription of going off-

shell with ν and $k\cdot k'$ fixed reduces to the usual ansatz of treating S waves as functions of μ only and P waves as functions of μ times $\vec{k}\cdot\vec{k}'$. At $\mu = m_\pi$, the g's reduce to scattering lengths and (3.5) reduces to a constant plus a piece proportional to $\vec{k}\cdot\vec{k}'$. From the latter, one can determine[29] the coefficient d_2 in (3.22).

D. Problems in the S waves

If we were to (mistakenly) conclude that the smallness of symmetric scattering length implies that the symmetric (+) amplitude has a small effect on the S waves then the natural low-energy expansion for an S wave would be

$$\mu q \cot\delta = \text{constant}, \tag{5.7}$$

where the factor of q is conventional and the factor of μ appears because the odd amplitude $A^{(-)} + \nu B^{(-)}$ is proportional to $\nu \approx \mu$. Experimentally,[30] the $I = \frac{3}{2}$ phase shift is well represented by Eq. (5.7), with the constant agreeing with that given by (5.3). For $I = \frac{1}{2}$, however, (5.7) completely disagrees with experiment.[30] It is qualitatively wrong for pion kinetic energies as low as 50 MeV. In part, this failure is due to the neglect of the even (+) amplitude which is accidentally small at threshold but comparable to the odd (−) amplitude away from threshold. This cannot be the whole story, however, since the (+) amplitude also contributes to the $I = \frac{3}{2}$ channel. At 50 MeV above threshold the (−) amplitude must have ceased to be linear in ν in just such a way as to compensate the energy dependence of the (+) amplitude in the $I = \frac{3}{2}$ channel.

The low-energy behavior of the π-N S waves does not seem to be of the simple type which can be adequately described by scattering lengths or expansions of invariant amplitudes. In realistic calculations of pion condensation, the S-wave interactions, although smaller than the P waves, often play an important role. To our knowledge none of the existing calculations (including our own) has gone beyond the approximation of Eq. (5.7) which is demonstrably wrong. Considerable work toward understanding π-N S waves and their role in condensation must be done before quantitative results can be claimed.

E. The P waves

The P-wave amplitudes are dominated by the Δ resonance. At threshold it produces a P-wave attraction of roughly the same size as the Born term and hence encourages condensation. In a fully developed pion condensation with $\mu \gtrsim m_\pi$ there will be a large contribution to the energy from the processes "condensed pion" $+ N \rightarrow \Delta$. This will lead to a large number of "virtual" Δ's and one must take account of the fact that they obey Fermi statistics. The easiest way to do this is to introduce the Δ as a separate elementary particle. This is easily accomplished in our general formalism and was illustrated in the preceding paper.

As a point of consistency, it has been shown that at neutron-star densities, the nucleon and Δ can be considered as two independent species of fermions.[31]

F. Summary

In previous sections we have stressed that a proper treatment of the correlation energy is required in a quantitative calculation. The point of this section has been that before a quantitative calculation can be done we must also understand the details of the low-energy π-N interaction. It is the S waves which are likely to cause trouble. The latter can probably be well approximated by introducing the Δ as a separate species. The confusing behavior of the S waves is what keeps us from asserting that (2.22) is a definitive formula for the Hartree energy E^H.

ACKNOWLEDGMENT

The hospitality of the Director and Faculty of the Institute for Advanced Study is gratefully acknowledged by D. K. C. and J. T. M. Another of us (R. F. D.) is grateful for the hospitality of Fermilab, where some of this work was done.

APPENDIX A: FEYNMAN INTEGRALS IN A MEDIUM AT $T=0$

As shown in Sec. II, in the presence of a medium, the conservation laws (isospin, baryon numbers, etc.) could be directly incorporated in the propagators through the substitution

$$\begin{aligned} &p_4 \rightarrow p_4 - i\mu \equiv \tilde{p}_4 \\ &(p_1, p_2, p_3) \rightarrow (p_1, p_2, p_3) \equiv (\tilde{p}_1, \tilde{p}_2, \tilde{p}_3)\,. \end{aligned} \tag{A1}$$

In this appendix we will review some well-known results of the literature[32] pertaining to

(a) methods for integral evaluations in the presence of a medium,

(b) list of the most important integrals,

(c) method of separation of the many-body effects from purely field-theoretical contributions.

1. Example

Evaluate

$$\int d^3p\, dp_4\, \frac{1}{(p_4 - i\mu)^2 + \vec{p}^2 + m^2} = L^{\mu \neq 0}(0, m^2)\,. \tag{A2}$$

The poles in p_4 are at

$$p_4^+ = i\mu + i(\vec{p}^2 + m^2)^{1/2}\,, \qquad p_4^- = i\mu - i(\vec{p}^2 + m^2)^{1/2}\,. \tag{A3}$$

The pole p_4^+ contribution to the integral will be the same for all values of μ, since $(p_4^+ - p_4^-)$ is μ-independent. The pole p_4^- will only contribute to the integral under the circumstances that $\mu - (\vec{p}^2 + m^2)^{1/2}$ is a positive number. We will express this by the function $\theta(\mu - (\vec{p}^2 + m^2)^{1/2})$ inserted with the (d^3p) integration. Consequently

$$\begin{aligned} L^{\mu\neq 0}(0,m^2) &= L^{\mu=0}(0,m^2) + \int d^3p\,\theta(\mu - (p^2+m^2)^{1/2})\left(-\frac{2\pi}{2(p^2+m^2)^{1/2}}\right) \\ &= L^{\mu=0}(0,m^2) - 4\pi^2 \int \frac{p^2\theta(\mu-(p^2+m^2)^{1/2})}{(p^2+m^2)^{1/2}}\,dp \\ &= L^{\mu=0}(0,m^2) - 2\pi^2\theta(\mu-m)\left[\mu(\mu^2-m^2)^{1/2} - m^2\ln\frac{\mu+(\mu^2-m^2)^{1/2}}{m}\right], \end{aligned} \tag{A4}$$

with

$$L^{\mu=0}(0,m^2) = 2\pi^2\left[\Lambda(\Lambda^2+m^2)^{1/2} - m^2\ln\frac{(\Lambda^2+m^2)^{1/2}+\Lambda}{m}\right], \tag{A5}$$

and where Λ is the ultraviolet cutoff for the momentum p. Notice that the second term on the right of Eq. (A4) could be deduced from the first through the substitution

$$\mu^2 - m^2 = \Lambda^2\,. \tag{A6}$$

In effect this example has shown us how to separate the field-theoretical effects from the many-body effects and, in addition, showed how the parameter $(\mu^2 - m^2)^{1/2}$ acts as a built-in ultraviolet cutoff to the many-body contribution of the integral.

2. List of some useful integrals

(a)
$$\begin{aligned} L^{\mu\neq 0}(k,m^2) &= \int \frac{d^4p}{(p_4 - i\mu)^2 + 2(p_4 - i\mu)k_4 + \vec{p}^2 + 2\vec{p}\cdot\vec{k} + m^2} \\ &= L^{\mu=0}(k,m^2) - 2\pi^2\theta(\mu - (m^2-k^2)^{1/2})\left[\mu(\mu^2 - m^2 + k^2)^{1/2} + (k^2-m^2)\ln\frac{\mu+(\mu^2-m^2+k^2)^{1/2}}{(m^2-k^2)^{1/2}}\right]; \end{aligned} \tag{A7}$$

(b)
$$\begin{aligned} L_\nu^{\mu\neq 0}(k,m^2) &= \int \frac{\tilde{p}_\nu\,d^4p}{\tilde{p}^2 + 2\tilde{p}k + m^2} \\ &= L_\nu^{\mu=0}(k,m^2) + 2\pi^2\theta(\mu-(m^2-k^2)^{1/2})\Big\{k_\nu\left[\mu(\mu^2-m^2+k^2)^{1/2} + (k^2-m^2)\ln\frac{\mu+(\mu^2-m^2+k^2)^{1/2}}{(m^2-k^2)^{1/2}}\right] \\ &\qquad + \tfrac{2}{3}i\delta_{\nu 4}(\mu^2 - m^2 + k^2)\Big\}\,; \end{aligned} \tag{A8}$$

(c)
$$\begin{aligned} Q^{\mu\neq 0}(k,m^2) &= \int \frac{d^4p}{(\tilde{p}^2 + 2\tilde{p}k + m^2)^2} \\ &= Q^{\mu=0}(k,m^2) - 2\pi^2\theta(\mu - (m^2-k^2)^{1/2})\ln\frac{\mu+(\mu^2-m^2+k^2)^{1/2}}{(m^2-k^2)^{1/2}}\,; \end{aligned} \tag{A9}$$

(d)
$$\begin{aligned} Q_\nu^{\mu\neq 0}(k,m^2) &= \int \frac{\tilde{p}_\nu\,d^4p}{(\tilde{p}^2 + 2\tilde{p}k + m^2)^2} \\ &= Q_\nu^{\mu=0}(k,m^2) + 2\pi^2\theta(\mu - (m^2-k^2)^{1/2})\left[k_\nu\ln\frac{\mu+(\mu^2-m^2+k^2)^{1/2}}{(m^2-k^2)^{1/2}} + i\delta_{\nu 4}(\mu^2-m^2+k^2)^{1/2}\right]; \end{aligned} \tag{A10}$$

(e) $$Q_{\rho\nu}^{\mu\neq 0}(k,m^2)=\int \frac{\tilde{p}_\rho \tilde{p}_\nu d^4p}{(\tilde{p}^2+2\tilde{p}k+m^2)^2}$$

$$=Q_{\rho\nu}^{\mu\neq 0}-2\pi^2\theta(\mu-(m^2-k^2)^{1/2})\left\{\frac{\delta_{\rho\nu}}{2}\left[\mu(\mu^2-m^2+k^2)^{1/2}+(k^2-m^2)\ln\frac{\mu+(\mu^2-m^2+k^2)^{1/2}}{(m^2-k^2)^{1/2}}\right]\right.$$

$$+k_\rho k_\nu \ln\frac{\mu+(\mu^2-m^2+k^2)^{1/2}}{(m^2-k^2)^{1/2}}$$

$$\left.+i(k_\nu\delta_{\rho 4}+k_\rho\delta_{\nu 4})(\mu^2-m^2+k^2)^{1/2}-\delta_{\nu 4}\delta_{\rho 4}\mu(\mu^2-m^2+k^2)^{1/2}\right\}. \tag{A11}$$

3. General considerations

As seen in the above examples, all the Feynman integrals in a medium do separate, and the purely many-body contribution has a built-in cutoff and is consequently always regular. This implies that the regularization of the free theory is sufficient in the presence of a medium, and the effects of the medium will be in altering the renormalization of the theory by finite amounts.

The generalization of the previous results could be simply achieved in that any diagram will consist of integrations over a product of denominators which are quadratic in p; then using Feynman's method

$$\frac{1}{a_1a_2\cdots a_n}=n!\int dx_1dx_2\cdots dx_n\,\delta(1-x_1-x_2-\cdots-x_n)\frac{1}{[a_1x_1+a_2x_2+\cdots+a_nx_n]^n},$$

where a_1 are quadratic in p, the right-hand side will be a quadratic term raised to the nth power, and on which the same manipulations, separating the contribution of p_4^+ and p_4^-, could be achieved.

APPENDIX B: SPONTANEOUS BREAKDOWN OF DISCRETE SYMMETRIES

In Sec. IV we discussed the implications of the "spontaneous breakdown," in the ground state with pion condensate, of the continuous symmetry associated with I_3 rotations. In this appendix we comment briefly on the obvious result that a state $|G'\rangle$ for which

$$\langle G'|\pi|G'\rangle\neq 0 \tag{B1}$$

cannot be an eigenstate of parity or time reversal. The original Lagrangian—defined in Eq. (2.2)—is invariant under these transformations; consequently P, CP, and T are also "spontaneously broken" by the presence of the pion condensate.[33]

The strength of the P and T violations can readily be estimated by referring to (3.6), from which one can see that our explicit P and T violating interaction will arise in second-order perturbation theory from terms of the form

$$\cos\theta\sin\theta(k_\alpha V_3^\alpha k_\beta A_2^\beta). \tag{B2}$$

Since $(k_\alpha)\approx O(m_\pi)$, this term is second order in the size of the chiral-symmetry breaking. For small condensate amplitude ($\theta\simeq 0$) the magnitude of the breaking is further suppressed by the factor $\cos\theta\sin\theta=\frac{1}{2}\sin(2\theta)$.

*Research sponsored in part by the U. S. Atomic Energy Commission, under Grant No. AT(11-1)-2220 and Contract No. W-7405-ENG-36.

†On leave from Physics Department, American University of Beirut, Beirut, Lebanon.

[1]D. Campbell, R. Dashen, and J. Manassah, preceding paper, Phys. Rev. D 12, 979 (1975).

[2]References to pion condensation are contained in I.

[3]B. W. Lee, *Chiral Dynamics* (Gordon and Breach, New York, 1972); S. Gasiorowicz and D. A. Geffen, Rev. Mod. Phys. 41, 331 (1969); V. de Alfaro *et al.*, *Currents in Hadron Physics* (North-Holland, Amsterdam, 1973); R. Dashen, in *Developments in High-Energy Physics*, proceedings of the International School of Physics Enrico Fermi, Course LIV, edited by R. Gatto (Academic, New York, 1973); R. Dashen and M. Weinstein, Phys. Rev. 183, 1261 (1969); S. Weinberg, in *Proceedings of the Fourteenth International Conference on High Energy Physics, Vienna, 1968*, edited by J. Prentki and J. Steinberger (CERN, Geneva, 1968); B. Zumino, in *Hadrons and Their Interactions*, proceedings of the International School of Physics "Ettore Majorana," edited by A. Zichichi (Academic, New York, 1968). These are reviews from which numerous further

references can be obtained.
[4]R. Dashen and J. Manassah, Phys. Lett. 47A, 453 (1974); 50B, 460 (1974).
[5]G. Jona-Lasinio, Nuovo Cimento 34, 1790 (1964); Y. Nambu, Phys. Lett. 26B, 626 (1968); S. Coleman and E. Weinberg, Phys. Rev. D 7, 1888 (1973); J. Cornwall, R. Jackiw, and E. Tomboulis, *ibid.* 10, 2428 (1974); R. Norton and J. Cornwall, Ann. Phys. (N.Y.) 91, 106 (1975). See also the reviews of Ref. 3.
[6]This use of chemical potentials is standard in statistical mechanics. Their use is illustrated in I.
[7]See especially S. Coleman, E. Weinberg, J. Cornwall, and R. Norton cited in Ref. 5.
[8]C.-K. Au, G. Baym, and E. Flowers, Phys. Lett. 51B, 1 (1974).
[9]The authors of Ref. 8 stress the importance of the analytic properties of G.
[10]It should be understood that since our Feynman rules depend on φ^{cl}, these "vacuum" graphs will not be zero after renormalization. They will be finite functionals of φ^{cl} which vanish only when $\varphi^{cl} = \langle\varphi\rangle_{vac}$.
[11]This may be seen by thinking of Fig. 3 as representing a particle-hole pair being created from and annihilating into the ground state.
[12]The reader is warned that our definitions of E^v, E^H, and E^c do not always correspond to the definitions of similarly named quantities appearing in the literature.
[13]E^v is not the energy of the vacuum unless $\varphi^{cl} = \langle\varphi\rangle_{vac}$, in which case it vanishes by convention.
[14]This is a dressed propagator in the presence of the field φ^{cl}. It contains all effects of φ^{cl} including those induced by radiative corrections. It does not contain many-body effects coming from the presence of occupied fermion states.
[15]This correlation energy can be thought of as a many-body correlation energy computed with the dressed propagator of Ref. 14 and using multibody interactions induced by radiative corrections as well as the basic vertices of the theory. The interaction vertices as well as the propagators depend on φ^{cl}. It is clear that E^c is complicated in general. In the simplest cases where only simple φ^{cl}-independent interactions occur and the propagators are approximated by free propagators in the field φ^{cl}, E^c reduces to the correlation energy familiar in the many-body problem.
[16]It has been suggested by T. D. Lee and G. C. Wick that in nuclear matter the energy required to change the chiral radius A can be compensated for by a decrease in the nucleon mass. This leads to a phase transition which is different from pion condensation and basically has nothing to do with chiral symmetry. See Appendix B of I for a summary and references.
[17]It would be of considerable interest to compare the correlation energy using a consistent chirally invariant approximation in, say, the σ model. Chiral symmetry *may* tend to make E^c smaller than one might expect.
[18]D. Gross and R. Jackiw, Nucl. Phys. B14, 269 (1969); D. Boulware and L. Brown, Phys. Rev. 156, 1724 (1967); R. Dashen and S. Lee, *ibid.* 187, 2017 (1969); and papers cited in Ref. 5.
[19]S. Weinberg, Phys. Rev. 166, 1568 (1968).
[20]R. Dashen, Phys. Rev. 183, 1245 (1969), or one of the reviews in Ref. 3.
[21]The usage of the symbol f_π is not uniform in the literature. The f_π defined by Eq. (3.10) is measured to be $f_\pi^2 \approx m_\pi^2/2$.
[22]T. Cheng and R. Dashen, Phys. Rev. Lett. 26, 594 (1971).
[23]H. Pilkuhn *et al.*, Nucl. Phys. B65, 460 (1973). ($\langle N|\Sigma|N\rangle$ is called $\sigma_{NN}^{\pi\pi}$ here.)
[24]This is implied by the way the nonlinear Lagrangians are constructed. See, for example, B. Lee cited in Ref. 3.
[25]R. Moorhouse, Annu. Rev. Nucl. Sci. 19, 301 (1969).
[26]S. Adler and R. Dashen, *Current Algebras and Applications to Particle Physics* (Benjamin, New York, 1968).
[27]G. Höhler, H. Jakob, and R. Strauss, Nucl. Phys. B39, 237 (1972).
[28]This is the "Higgs mechanism." P. Higgs, Phys. Lett. 12, 132 (1964); Phys. Rev. Lett. 13, 508 (1964); Phys. Rev. 145, 1156 (1966).
[29]The P-wave scattering length can be found in Ref. 23. The value of d_2 computed this way agrees reasonably well with that obtained from Ref. 27.
[30]J. Carter, D. Bugg, and A. Carter, Nucl. Phys. B58, 378 (1973).
[31]R. Dashen and R. Rajaraman, Phys. Rev. D 10, 694 (1974); 10, 708 (1974).
[32]A. A. Abrikosov, L. P. Gorkov, and I. Ye. Dzyaloshinskii, *Quantum Field Theoretical Methods in Statistical Physics* (Pergamon, New York, 1965).
[33]T. D. Lee, Phys. Rev. D 8, 1226 (1973).